

# A note on geometric characterization of quantum isometries of classical manifolds

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## Abstract

If a compact quantum group acts isometrically on a (possibly disconnected) compact smooth Riemannian manifold  $M$  such that the action (say  $\alpha$ ) commutes with the Laplacian, i.e. isometric in the sense of [12] then it is known ([12]) that the ‘differential’ of the action preserves Riemannian inner product on forms in the sense that  $\langle\langle (d \otimes \text{id})(\alpha(f)), (d \otimes \text{id})(\alpha(g)) \rangle\rangle = \alpha(\langle\langle df, dg \rangle\rangle)$  for all smooth functions  $f, g$ , where  $\langle\langle \cdot, \cdot \rangle\rangle$  denotes the Riemannian inner-product viewed as a  $C^\infty(M)$ -valued inner-product on the bimodule of one-forms. In this note, we prove a partial converse to this, under the additional assumption that  $M$  is oriented and the action preserves the orientation in a suitable sense. Using this, an alternative line of arguments is given for the main result of [12].

## 1 Introduction

It is a very important and interesting problem in the theory of quantum groups and noncommutative geometry to study ‘quantum symmetries’ of various classical and quantum structures. Indeed, symmetries of physical systems (classical or quantum) were conventionally modelled by group actions, and after the advent of quantum groups, group symmetries were naturally generalized to symmetries given by quantum group action. In this context, it is natural to think of quantum automorphism or the full quantum symmetry groups of various mathematical and physical structures. The underlying basic principle of defining a quantum automorphism group of a given mathematical structure consists of two steps : first, to identify (if possible) the group of automorphisms of the structure as a universal object in a suitable category, and then, try to look for the universal object in a similar but bigger category by replacing groups by quantum groups of appropriate type. The formulation and study of such quantum symmetries in terms of universal Hopf algebras date back to Manin [19]. In the analytic set-up of compact quantum groups, it was considered by S. Wang who defined and studied quantum permutation groups of finite sets and quantum automorphism groups of finite dimensional algebras, such questions were taken up by a number of mathematicians including Banica, Bichon (see, e.g. [2], [3], [9], [24]), and more recently in the framework of Connes’ noncommutative geometry ([11]) by Goswami, Bhowmick, Skalski, Banica and others who have extensively studied

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the quantum group of isometries (or quantum isometry group) defined in [13] (see also [8], [6], [4] etc.). In this context, it is important to compute such quantum isometry groups for classical (compact) Riemannian manifolds.

In the classical case, i.e. smooth group-actions on a Riemannian manifold  $M$ , the action commutes with the Hodge Laplacian  $-d^*d$  if and only if its differential is an isometry between co-tangent spaces for the given Riemannian structure, i.e. preserves the  $C^\infty(M)$  valued inner product on the bimodule of smooth one-forms. It is natural to see whether this extends to the quantum case. This is the aim of the present article. Indeed, it is easy to see one-way: if a compact quantum group action commutes with the Laplacian (this is what is termed as ‘isometric action’ in [13]), then it is smooth and preserves the inner product. However, we have been able to prove the converse only in a slightly restricted set-up, namely when the manifold is oriented and the action also preserves the orientation in a suitable sense.

## 2 Notations and preliminaries

We follow the notations and set-up of [12], which we briefly recall here. All the Hilbert spaces are over  $\mathbb{C}$  unless mentioned otherwise. If  $V$  is a vector space over real numbers we denote its complexification by  $V_{\mathbb{C}}$ . For a vector space  $V$ ,  $V'$  stands for its algebraic dual. For a  $C^*$  algebra  $\mathcal{C}$ ,  $\mathcal{M}(\mathcal{C})$  will denote its multiplier algebra.  $\oplus$  and  $\otimes$  will denote the algebraic direct sum and algebraic tensor product respectively. We shall denote the  $C^*$  algebra of bounded operators on a Hilbert space  $\mathcal{H}$  by  $\mathcal{B}(\mathcal{H})$  and the  $C^*$  algebra of compact operators on  $\mathcal{H}$  by  $\mathcal{B}_0(\mathcal{H})$ .  $Sp$  ( $\overline{Sp}$ ) stands for the linear span (closed linear span). Also WOT and SOT for the weak operator topology and the strong operator topology respectively. Let  $\mathcal{C}$  be an algebra. Then  $\sigma_{ij} : \underbrace{\mathcal{C} \otimes \mathcal{C} \otimes \dots \otimes \mathcal{C}}_{n\text{-times}} \rightarrow \underbrace{\mathcal{C} \otimes \mathcal{C} \otimes \dots \otimes \mathcal{C}}_{n\text{-times}}$  is the

flip map between  $i$  and  $j$ -th place and  $m_{ij} : \underbrace{\mathcal{C} \otimes \mathcal{C} \otimes \dots \otimes \mathcal{C}}_{n\text{-times}} \rightarrow \underbrace{\mathcal{C} \otimes \mathcal{C} \otimes \dots \otimes \mathcal{C}}_{(n-1)\text{-times}}$  is

the map obtained by multiplying  $i$  and  $j$ -th entry. In case we have two copies of an algebra we shall simply denote by  $\sigma$  and  $m$  for the flip and multiplication map respectively. We shall need several types of topological tensor products in this paper:  $\hat{\otimes}$ ,  $\bar{\otimes}$ ,  $\bar{\otimes}_{in}$  (to be explained in subsequent sections). Also for a Hilbert space  $\mathcal{H}$  and a  $C^*$  algebra or a locally convex  $*$  algebra  $\mathcal{C}$  we shall consider the trivial Hilbert (bi)module  $\mathcal{H} \bar{\otimes} \mathcal{C}$  with the obvious right and left action of  $\mathcal{C}$  on  $\mathcal{H} \bar{\otimes} \mathcal{C}$  coming from algebra multiplication of  $\mathcal{C}$  and obvious  $\mathcal{C}$  valued inner product. When  $\mathcal{H} = \mathbb{C}^N$ , the (bi)module is called the trivial  $\mathcal{C}$  (bi)module of rank  $N$ . Usually, we use  $\langle, \rangle$  and  $\langle\langle, \rangle\rangle$  for the scalar valued inner product (of a Hilbert space) and the algebra-valued inner product (of a Hilbert bimodule) respectively. For a Hopf algebra  $H$ , for any  $\mathbb{C}$ -linear map  $f : H \rightarrow H \otimes H$ , we write  $f(q) = q_{(1)} \otimes q_{(2)}$  (Sweedler’s notation). For an algebra or module  $\mathcal{A}$  and a  $\mathbb{C}$ -linear map  $\Gamma : \mathcal{A} \rightarrow \mathcal{A} \otimes H$ , we shall also use an analogue of Sweedler’s notation and write  $\Gamma(a) = a_{(0)} \otimes a_{(1)}$ .

We begin by recalling from [23] the tensor product of two  $C^*$  algebras  $\mathcal{C}_1$  and  $\mathcal{C}_2$  and let us choose the minimal or spatial tensor product between two  $C^*$  algebras. The corresponding  $C^*$  algebra will be denoted by  $\mathcal{C}_1 \hat{\otimes} \mathcal{C}_2$  throughout this paper. However we need to consider more general topological spaces and algebras, namely, locally convex  $*$ -algebras embedded in  $C^*$  algebras with topology given by countable family seminorms coming from closed derivations, which is a special class of  $C^*$ -normed smooth algebras in the sense of Blackadar and Cuntz ([10]).

**Definition 2.1** *A unital Fréchet  $*$  algebra  $\mathcal{A}$  will be called a ‘nice’ algebra if there is a  $C^*$ -norm  $\|\cdot\|$  on  $\mathcal{A}$  and the underlying locally convex topology of  $\mathcal{A}$  comes from the family of seminorms  $\{\|\cdot\|_{\underline{\alpha}}\}$  given below:  
 $\{\|x\|_{\underline{\alpha}} = \|\delta_{\underline{\alpha}}(x)\|\}$ , where  $\underline{\alpha} = (i_1, \dots, i_k) : 1 \leq i_j \leq k, k \geq 1$  is a multi index or  $\underline{\alpha} = \phi(\text{null index})$ ,  $\delta_{\underline{\alpha}} = \delta_{i_1} \dots \delta_{i_k}$ ,  $\delta_{\phi} = \text{id}$  and where each  $\delta_i$  denotes a  $\|\cdot\|$ -closable  $*$ -derivation from  $\mathcal{A}$  to itself.*

Given two such ‘nice’ algebras  $\mathcal{A}(\subset \mathcal{A}_1)$  and  $\mathcal{B}(\subset \mathcal{B}_1)$ , where  $\mathcal{A}_1, \mathcal{B}_1$  denote respectively the  $C^*$ -completion of  $\mathcal{A}, \mathcal{B}$  in the corresponding  $C^*$ -norms, we choose the injective tensor product norm on  $\mathcal{A} \otimes \mathcal{B}$ , i.e. view it as a dense subalgebra of  $\mathcal{A}_1 \hat{\otimes} \mathcal{B}_1$ .  $\mathcal{A} \otimes \mathcal{B}$  has natural (closable  $*$ ) derivations of the forms  $\tilde{\delta} = \delta \otimes \text{id}$  as well as  $\tilde{\eta} = \text{id} \otimes \eta$  where  $\delta, \eta$  are closable  $*$ -derivations on  $\mathcal{A}$  and  $\mathcal{B}$  respectively. Clearly,  $\tilde{\delta}$  commute with  $\tilde{\eta}$ . We topologize  $\mathcal{A} \otimes \mathcal{B}$  by the family of seminorms coming from such derivations, i.e.  $\{\|\cdot\|_{\underline{\alpha}\underline{\beta}}\}$  where  $\|\cdot\|$  is the injective  $C^*$ -norm and

$$\|X\|_{\underline{\alpha}\underline{\beta}} = \|\tilde{\delta}_{\underline{\alpha}} \tilde{\eta}_{\underline{\beta}}(X)\|,$$

$\underline{\alpha} = (i_1, \dots, i_k)$ ,  $\underline{\beta} = (j_1, \dots, j_l)$  some multi-indices as before and  $\delta_i, \eta_j$ ’s being closable  $*$ -derivations on  $\mathcal{A}$  and  $\mathcal{B}$  respectively.

We refer to [12] and references therein (in particular [10]) for discussion on such algebras. We actually need algebras of the form  $C^\infty(M) \hat{\otimes} \mathcal{A}$  where  $M$  a smooth compact manifold possibly with boundary and  $\mathcal{A}$  is a unital  $C^*$  algebra. By the nuclearity of  $C^\infty(M)$  as a locally convex space the above tensor product is isomorphic with  $C^\infty(M, \mathcal{A})$  with the Fréchet topology coming from the smooth vector fields of  $M$ . Moreover, it is proved in [12] that  $C^\infty(M, \mathcal{A})$  is again a nice algebra and we also have Fréchet  $*$ -algebra isomorphism between  $C^\infty(M, \mathcal{A}) \hat{\otimes} \mathcal{B}$  and  $C^\infty(M, \mathcal{A} \hat{\otimes} \mathcal{B})$ , where  $\mathcal{B}$  is any other unital  $C^*$  algebra.

We also need Hilbert bimodules over such algebras and their internal and external tensor products. Let  $\mathcal{E}_1$  and  $\mathcal{E}_2$  be two Hilbert bimodules over two locally convex  $*$  algebras(nice)  $\mathcal{C}_1$  and  $\mathcal{C}_2$  respectively. We denote the algebra valued inner product for the Hilbert bimodules by  $\langle\langle \cdot, \cdot \rangle\rangle$ . When the bimodule is a Hilbert space, we denote the corresponding scalar valued inner product by  $\langle \cdot, \cdot \rangle$ . Then  $\mathcal{E}_1 \otimes \mathcal{E}_2$  has an obvious  $\mathcal{C}_1 \otimes \mathcal{C}_2$  bimodule structure, given by  $(a \otimes b)(e_1 \otimes e_2)(a' \otimes b') = ae_1a' \otimes be_2b'$  for  $a, a' \in \mathcal{C}_1, b, b' \in \mathcal{C}_2$  and  $e_1 \in \mathcal{E}_1, e_2 \in \mathcal{E}_2$ . Also define  $\mathcal{C}_1 \otimes \mathcal{C}_2$  valued inner product by  $\langle\langle e_1 \otimes e_2, f_1 \otimes f_2 \rangle\rangle = \langle\langle e_1, f_1 \rangle\rangle \otimes \langle\langle e_2, f_2 \rangle\rangle$  for  $e_1, f_1 \in \mathcal{E}_1$  and  $e_2, f_2 \in \mathcal{E}_2$ . We denote the completed module by  $\mathcal{E}_1 \bar{\otimes} \mathcal{E}_2$ . In fact  $\mathcal{E}_1 \bar{\otimes} \mathcal{E}_2$  is an  $\mathcal{C}_1 \hat{\otimes} \mathcal{C}_2$  bimodule. This is called the exterior tensor product of two bimodules. In particular if one of the

bimodule is a Hilbert space  $\mathcal{H}$  (bimodule over  $\mathbb{C}$ ) and the other is a  $C^*$  algebra  $\mathcal{Q}$  (bimodule over itself), then the exterior tensor product gives the usual Hilbert  $\mathcal{Q}$  module  $\mathcal{H} \bar{\otimes} \mathcal{Q}$ . When  $\mathcal{H} = \mathbb{C}^N$ , we have a natural identification of an element  $T = ((T_{ij})) \in M_N(\mathcal{Q})$  with the right  $\mathcal{Q}$  linear map of  $\mathbb{C}^N \otimes \mathcal{Q}$  given by

$$e_i \mapsto \sum e_j \otimes T_{ji},$$

where  $\{e_i\}_{i=1,\dots,N}$  is a basis for  $\mathbb{C}^N$ . We can take tensor products of maps between two Hilbert bimodules under. We shall need such tensor product of maps, which are ‘isometric’ in some sense. Let  $T_i : \mathcal{E}_i \rightarrow \mathcal{F}_i$ ,  $i = 1, 2$  be two  $\mathbb{C}$ -linear maps and  $\mathcal{E}_i, \mathcal{F}_i$  be Hilbert bimodules over  $\mathcal{C}_i, \mathcal{D}_i$  ( $i = 1, 2$ ) respectively. Moreover, suppose that  $\langle\langle T_i(\xi_i), T_i(\eta_i) \rangle\rangle = \alpha_i \langle\langle \xi_i, \eta_i \rangle\rangle$ ,  $\xi_i, \eta_i \in \mathcal{E}_i$  where  $\alpha_i : \mathcal{C}_i \rightarrow \mathcal{D}_i$  are  $*$ -homomorphisms. Then it is easy to show that the algebraic tensor product  $T := T_1 \otimes_{alg} T_2$  also satisfies  $\langle\langle T(\xi), T(\eta) \rangle\rangle = (\alpha_1 \otimes \alpha_2)(\langle\langle \xi, \eta \rangle\rangle)$  and hence extends to a well defined continuous map from  $\mathcal{E}_1 \bar{\otimes} \mathcal{E}_2$  to  $\mathcal{F}_1 \bar{\otimes} \mathcal{F}_2$  again to be denoted by  $T_1 \otimes T_2$ .

Let  $\mathcal{B}, \mathcal{C}, \mathcal{D}$  be three locally convex  $*$  algebras. Also let  $\mathcal{E}_1$  be an  $\mathcal{B} - \mathcal{C}$  Hilbert bimodule and  $\mathcal{E}_2$  be a  $\mathcal{C} - \mathcal{D}$  Hilbert bimodule. Then  $\mathcal{E}_1 \otimes_{\mathcal{C}} \mathcal{E}_2$  is an  $\mathcal{B} - \mathcal{D}$  bimodule in the usual way. We can define a  $\mathcal{D}$  valued inner product that will make  $\mathcal{E}_1 \otimes_{\mathcal{C}} \mathcal{E}_2$  a pre-Hilbert  $\mathcal{B} - \mathcal{D}$  bimodule. For that take  $\omega_1, \omega_2 \in \mathcal{E}_1$  and  $\eta_1, \eta_2 \in \mathcal{E}_2$  and define

$$\langle\langle \omega_1 \otimes \eta_1, \omega_2 \otimes \eta_2 \rangle\rangle := \langle\langle \eta_1, \langle\langle \omega_1, \omega_2 \rangle\rangle \eta_2 \rangle\rangle.$$

Let  $\mathcal{I} = \{\xi \in \mathcal{E}_1 \otimes_{\mathcal{C}} \mathcal{E}_2 \text{ such that } \langle\langle \xi, \xi \rangle\rangle = 0\}$ . Then define  $\mathcal{E}_1 \otimes_{in} \mathcal{E}_2 = \mathcal{E}_1 \otimes_{\mathcal{C}} \mathcal{E}_2 / \mathcal{I}$ . We note that this semi inner product is actually an inner product, so that  $\mathcal{I} = \{0\}$  (see proposition 4.5 of [15]). The topological completion of  $\mathcal{E}_1 \otimes_{in} \mathcal{E}_2$  is called the interior tensor product and we shall denote it by  $\mathcal{E}_1 \bar{\otimes}_{in} \mathcal{E}_2$ . We denote the projection map from  $\mathcal{E}_1 \otimes_{\mathcal{C}} \mathcal{E}_2$  to  $\mathcal{E}_1 \otimes_{in} \mathcal{E}_2$  by  $\pi$ . We also make the convention of calling a Hilbert  $\mathcal{A} - \mathcal{A}$  bimodule simply Hilbert  $\mathcal{A}$  bimodule.

### 3 Compact quantum groups, their representations and actions

**Definition 3.1** *A compact quantum group (CQG for short) is a unital  $C^*$  algebra  $\mathcal{Q}$  with a coassociative coproduct (see [17])  $\Delta$  from  $\mathcal{Q}$  to  $\mathcal{Q} \hat{\otimes} \mathcal{Q}$  such that each of the linear spans of  $\Delta(\mathcal{Q})(\mathcal{Q} \otimes 1)$  and that of  $\Delta(\mathcal{Q})(1 \otimes \mathcal{Q})$  is norm-dense in  $\mathcal{Q} \hat{\otimes} \mathcal{Q}$ .*

From this condition, one can obtain a canonical dense unital  $*$ -subalgebra  $\mathcal{Q}_0$  of  $\mathcal{Q}$  on which linear maps  $\kappa$  and  $\epsilon$  (called the antipode and the counit respectively) are defined making the above subalgebra a Hopf  $*$  algebra. In fact, this is the algebra generated by the ‘matrix coefficients’ of the (finite dimensional) irreducible non degenerate representations (to be defined shortly) of the CQG.

The antipode is an anti-homomorphism and also satisfies  $\kappa(a^*) = (\kappa^{-1}(a))^*$  for  $a \in \mathcal{Q}_0$ .

Let  $\mathcal{H}$  be a Hilbert space. Consider the multiplier algebra  $\mathcal{M}(\mathcal{B}_0(\mathcal{H}) \hat{\otimes} \mathcal{Q})$ . This algebra has two natural embeddings into  $\mathcal{M}(\mathcal{B}_0(\mathcal{H}) \hat{\otimes} \mathcal{Q} \hat{\otimes} \mathcal{Q})$ . The first one is obtained by extending the map  $x \mapsto x \otimes 1$ . The second one is obtained by composing this map with the flip on the last two factors. We will write  $w^{12}$  and  $w^{13}$  for the images of an element  $w \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \hat{\otimes} \mathcal{Q})$  by these two maps respectively. Note that if  $\mathcal{H}$  is finite dimensional then  $\mathcal{M}(\mathcal{B}_0(\mathcal{H}) \hat{\otimes} \mathcal{Q})$  is isomorphic to  $\mathcal{B}(\mathcal{H}) \otimes \mathcal{Q}$  (we don't need any topological completion).

**Definition 3.2** Let  $(\mathcal{Q}, \Delta)$  be a CQG. A unitary representation of  $\mathcal{Q}$  on a Hilbert space  $\mathcal{H}$  is a unitary element  $\tilde{U} \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \hat{\otimes} \mathcal{Q})$  such that  $(\text{id} \otimes \Delta)\tilde{U} = \tilde{U}^{12}\tilde{U}^{13}$ .

Now Let  $\mathcal{C}$  be a nice algebra and  $\mathcal{Q}$  be a compact quantum group. Then we have the following

**Definition 3.3** A  $\mathbb{C}$  linear map  $\alpha : \mathcal{C} \rightarrow \mathcal{C} \hat{\otimes} \mathcal{Q}$  is said to be a topological action of  $\mathcal{Q}$  on  $\mathcal{C}$  if

1.  $\alpha$  is a continuous  $*$  algebra homomorphism.
2.  $(\alpha \otimes \text{id})\alpha = (\text{id} \otimes \Delta)\alpha$  (co-associativity).
3.  $\text{Sp } \alpha(\mathcal{C})(1 \otimes \mathcal{Q})$  is dense in  $\mathcal{C} \hat{\otimes} \mathcal{Q}$  in the corresponding Fréchet topology.

Given a topological action  $\alpha$ , proceeding along the lines of [22], we can prove the existence of a maximal dense  $*$  subalgebra  $\mathcal{C}_0$  of  $\mathcal{C}$  such that the action is algebraic over  $\mathcal{C}_0$  in the sense that  $\alpha(\mathcal{C}_0) \subset \mathcal{C}_0 \otimes \mathcal{Q}_0$  and  $\text{Sp } \alpha(\mathcal{C}_0)(1 \otimes \mathcal{Q}_0) = \mathcal{C}_0 \otimes \mathcal{Q}_0$ . Note that if the Fréchet algebra is a  $C^*$  algebra, then the definition of a topological action coincides with the usual  $C^*$  action of a compact quantum group.

**Definition 3.4** A topological action  $\alpha$  is said to be faithful if the  $*$ -subalgebra of  $\mathcal{Q}$  generated by the elements of the form  $(\omega \otimes \text{id})\alpha$ , where  $\omega$  is a continuous linear functional on  $\mathcal{C}$ , is dense in  $\mathcal{Q}$ .

We now generalize the notion of unitary representation on Hilbert spaces to on Hilbert bimodules over nice, unital topological  $*$ -algebras. Let  $\mathcal{E}$  be a Hilbert  $\mathcal{C} - \mathcal{D}$  bimodule over topological  $*$ -algebras  $\mathcal{C}$  and  $\mathcal{D}$  and let  $\mathcal{Q}$  be a compact quantum group. If we consider  $\mathcal{Q}$  as a bimodule over itself, then we can form the exterior tensor product  $\mathcal{E} \bar{\otimes} \mathcal{Q}$  which is a  $\mathcal{C} \hat{\otimes} \mathcal{Q} - \mathcal{D} \hat{\otimes} \mathcal{Q}$  bimodule. Also let  $\alpha_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C} \hat{\otimes} \mathcal{Q}$  and  $\alpha_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D} \hat{\otimes} \mathcal{Q}$  be topological actions of  $\mathcal{C}$  and  $\mathcal{D}$  on  $\mathcal{Q}$  in the sense discussed earlier. Then using  $\alpha$  we can give  $\mathcal{E} \bar{\otimes} \mathcal{Q}$  a  $\mathcal{C} - \mathcal{D}$  bimodule structure given by  $a \cdot \eta \cdot a' = \alpha_{\mathcal{C}}(a) \eta \alpha_{\mathcal{D}}(a')$ , for  $\eta \in \mathcal{E} \bar{\otimes} \mathcal{Q}$  and  $a \in \mathcal{C}, a' \in \mathcal{D}$  (but without any  $\mathcal{D}$  valued inner product).

**Definition 3.5** A  $\mathbb{C}$ -linear map  $\Gamma : \mathcal{E} \rightarrow \mathcal{E} \bar{\otimes} \mathcal{Q}$  is said to be an  $\alpha_{\mathcal{D}}$  equivariant unitary representation of  $\mathcal{Q}$  on  $\mathcal{E}$  if

1.  $\Gamma(\xi d) = \Gamma(\xi) \alpha_{\mathcal{D}}(d)$  and  $\Gamma(c \xi) = \alpha_{\mathcal{C}}(c) \Gamma(\xi)$  for  $c \in \mathcal{C}, d \in \mathcal{D}$ .

2.  $\langle\langle \Gamma(\xi), \Gamma(\xi') \rangle\rangle = \alpha_{\mathcal{D}}(\langle\langle \xi, \xi' \rangle\rangle)$ , for  $\xi, \xi' \in \mathcal{E}$ .
3.  $(\Gamma \otimes \text{id})\Gamma = (\text{id} \otimes \Delta)\Gamma$  (co associativity)
4.  $\overline{Sp} \Gamma(\mathcal{E})(1 \otimes \mathcal{Q}) = \mathcal{E} \bar{\otimes} \mathcal{Q}$  (non degeneracy).

In the definition note that condition (2) allows one to define  $(\Gamma \otimes \text{id})$ . We recall some relevant results from [12].

**Lemma 3.6** *Let  $\mathcal{E}_1$  be a Hilbert  $\mathcal{B} - \mathcal{C}$  bimodule and  $\mathcal{E}_2$  be a Hilbert  $\mathcal{C} - \mathcal{D}$  bimodule.  $\alpha_{\mathcal{B}}, \alpha_{\mathcal{C}}, \alpha_{\mathcal{D}}$  be topological actions on a compact quantum group  $\mathcal{Q}$  of topological  $*$ -algebras  $\mathcal{B}, \mathcal{C}, \mathcal{D}$  respectively.  $\Gamma_1 : \mathcal{E}_1 \rightarrow \mathcal{E}_1 \bar{\otimes} \mathcal{Q}$  and  $\Gamma_2 : \mathcal{E}_2 \rightarrow \mathcal{E}_2 \bar{\otimes} \mathcal{Q}$  be  $\alpha_{\mathcal{C}}$  and  $\alpha_{\mathcal{D}}$  equivariant unitary representations as discussed earlier. Then*

$$\langle\langle \Gamma_2(\eta), \langle\langle \Gamma_1(\omega), \Gamma_1(\omega') \rangle\rangle \Gamma_2(\eta') \rangle\rangle = \alpha_{\mathcal{D}} \langle\langle \eta, \langle\langle \omega, \omega' \rangle\rangle \eta' \rangle\rangle .$$

**Lemma 3.7** *Let  $\mathcal{E}_1, \mathcal{E}_2, \mathcal{B}, \mathcal{C}, \mathcal{D}, \alpha_{\mathcal{B}}, \alpha_{\mathcal{C}}, \alpha_{\mathcal{D}}, \Gamma_1, \Gamma_2, \mathcal{Q}$  be as in Lemma 3.6. Then we have an  $\alpha_{\mathcal{D}}$  equivariant representation  $\Gamma$  of  $\mathcal{Q}$  on the Hilbert  $\mathcal{B} - \mathcal{D}$  bimodule  $\mathcal{E}_1 \bar{\otimes}_{in} \mathcal{E}_2$ .*

In particular when  $\mathcal{E}$  is the trivial  $\mathcal{C}$ -bimodule of rank  $N$ , we have the following:

**Lemma 3.8** *Given an  $\alpha$  equivariant representation  $\Gamma$  of  $\mathcal{Q}$  on  $\mathbb{C}^N \otimes \mathcal{C}$  such that  $\Gamma(e_i \otimes 1_{\mathcal{C}}) = \sum_{j=1}^N e_j \otimes b_{ji}$ ,  $b_{ij} \in \mathcal{C} \hat{\otimes} \mathcal{Q}$  for all  $i, j = 1, \dots, N$ , where  $\{e_i; i = 1, \dots, N\}$  is an orthonormal basis of  $\mathbb{C}^N$ , then  $U = ((b_{ij}))_{i,j=1,\dots,N}$  is a unitary element of  $M_N(\mathcal{C} \hat{\otimes} \mathcal{Q})$ .*

## 4 Locally convex $*$ algebras and Hilbert bimodules coming from classical geometry

Let  $M$  be a compact, Riemannian manifold. As in [12], we denote the algebra of real (complex) valued smooth functions on  $M$  by  $C^\infty(M)_{\mathbb{R}}$  ( $C^\infty(M)$ ). Clearly  $C^\infty(M)$  is the complexification of  $C^\infty(M)_{\mathbb{R}}$ . It is a nice algebra whose locally convex topology is given by a complete set of vector fields  $\{\delta_1, \dots, \delta_N\}$ . For details of this topology we refer the reader to Subsection 5.1 of [12]. With this locally convex topology in fact  $C^\infty(M)$  is a nice nuclear algebra so that we can consider topological action of  $\mathcal{Q}$  on  $C^\infty(M)$ . Also let  $\Lambda^k(C^\infty(M))$  be the space of smooth  $k$  forms on the manifold  $M$ . We equip  $\Lambda^1(C^\infty(M))$  with the natural locally convex topology induced by the locally convex topology of  $C^\infty(M)$  given by a family of seminorms  $\{p_{(U, (x_1, \dots, x_n), K, \beta)}\}$ , where  $(U, (x_1, \dots, x_n))$  is a local coordinate chart,  $\beta = (\beta_1, \beta_2, \dots, \beta_r)$  is a multi-index with  $\beta_i \in \{1, 2, \dots, n\}$  as before,  $K$  is a compact subset, and  $p_{(U, (x_1, \dots, x_n), K, \beta)}(\omega) := \sup_{x \in K, 1 \leq i \leq n} |\partial_{\beta} f_i(x)|$ , where  $f_i \in C^\infty(M)$  such that  $\omega|_U = \sum_{i=1}^n f_i dx_i|_U$ . It is clear from the definition that the differential map  $d : C^\infty(M) \rightarrow \Omega^1(C^\infty(M))$  is Fréchet continuous.

Now for a  $C^*$  algebra  $\mathcal{Q}$ ,  $\Lambda^k(C^\infty(M)) \bar{\otimes} \mathcal{Q}$  has a natural  $C^\infty(M) \hat{\otimes} \mathcal{Q}$  bimodule structure. The left action is given by

$$(\sum_i f_i \otimes q_i) (\sum_j [\pi_{(k)}(\omega_j)] \otimes q'_j) = (\sum_{i,j} [\pi_{(k)}(f_i \omega_j)] \otimes q_i q'_j)$$

The right action is similarly given. The inner product is given by

$$\langle\langle \sum_i \omega_i \otimes q_i, \sum_j \omega'_j \otimes q'_j \rangle\rangle = \sum_{i,j} \langle\langle \omega_i, \omega'_j \rangle\rangle \otimes q_i^* q'_j.$$

Topology on  $\Lambda^k(C^\infty(M)) \bar{\otimes} \mathcal{Q}$  is given by requiring  $\omega_n \rightarrow \omega$  if and only if  $\langle\langle \omega_n - \omega, \omega_n - \omega \rangle\rangle \rightarrow 0$  in  $C^\infty(M) \hat{\otimes} \mathcal{Q}$  or  $C^\infty(M, \mathcal{Q})$ .

## 5 Hodge $\star$ map

Now consider the case when  $M$  is orientable and a globally non-vanishing  $n$ -form ( $n$  being the dimension on  $M$ ) has been chosen. We introduce the Hodge star operator, which is a pointwise isometry  $* = *_x : \Lambda^k T_x^* M \rightarrow \Lambda^{n-k} T_x^* M$ . Choose a positively oriented orthonormal basis  $\{\theta^1, \theta^2, \dots, \theta^n\}$  of  $T_x^* M$ . Since  $*$  is a linear transformation it is enough to define  $*$  on a basis element  $\theta^{i_1} \wedge \theta^{i_2} \wedge \dots \wedge \theta^{i_k}$  ( $i_1 < i_2 < \dots < i_k$ ) of  $\Lambda^k T_x^* M$ . Note that

$$\begin{aligned} \text{dvol}(x) &= \sqrt{\det(\langle \theta^i, \theta^j \rangle)} \theta^1 \wedge \theta^2 \wedge \dots \wedge \theta^n \\ &= \theta^1 \wedge \theta^2 \wedge \dots \wedge \theta^n \end{aligned}$$

**Definition 5.1**  $*(\theta^{i_1} \wedge \theta^{i_2} \wedge \dots \wedge \theta^{i_k}) = \theta^{j_1} \wedge \theta^{j_2} \wedge \dots \wedge \theta^{j_{n-k}}$  where  $\theta^{i_1} \wedge \theta^{i_2} \wedge \dots \wedge \theta^{i_k} \wedge \theta^{j_1} \wedge \theta^{j_2} \wedge \dots \wedge \theta^{j_{n-k}} = \text{dvol}(x)$ .

Since we are using  $\mathbb{C}$  as the scalar field, we would like to define  $\bar{\omega}$  for a  $k$  form  $\omega$ . In the set-up introduced just before the definition we have some scalars  $c_{i_1, \dots, i_k}$  such that  $\omega(x) = \sum c_{i_1, \dots, i_k} \theta^{i_1} \wedge \theta^{i_2} \wedge \dots \wedge \theta^{i_k}$ . Then define  $\bar{\omega}$  to be  $\bar{\omega}(x) = \sum \bar{c}_{i_1, \dots, i_k} \theta^{i_1} \wedge \theta^{i_2} \wedge \dots \wedge \theta^{i_k}$ . Then the equation  $\langle\langle \omega, \eta \rangle\rangle = *(\bar{\omega} \wedge *\eta)$  defines an inner product on the Hilbert module  $\Lambda^k(C^\infty(M))$  for all  $k = 1, \dots, n$  which is the same as the  $C^\infty(M)$  valued inner product defined earlier. Then the Hodge star operator is a unitary between two Hilbert modules  $\Lambda^k(C^\infty(M))$  and  $\Lambda^{n-k}(C^\infty(M))$  i.e.  $\langle\langle *\omega, *\eta \rangle\rangle = \langle\langle \omega, \eta \rangle\rangle$ . Also for  $\omega, \eta \in \Lambda^k(C^\infty(M))$ , we have  $*\omega \wedge \eta = \langle\langle \bar{\omega}, \eta \rangle\rangle \text{dvol}$ . For further details about the Hodge star operator we refer the reader to [20].

Hence we have

$$(* \otimes \text{id}) : \Lambda^k(C^\infty(M)) \otimes \mathcal{Q} \rightarrow \Lambda^{n-k}(C^\infty(M)) \otimes \mathcal{Q}.$$

Since Hodge  $*$  operator is an isometry,  $(* \otimes \text{id})$  is continuous with respect to the Hilbert module structure of  $\dot{\Lambda}(C^\infty(M)) \bar{\otimes} \mathcal{Q}$ . So we have

$$(* \otimes \text{id}) : \Lambda^k(C^\infty(M)) \bar{\otimes} \mathcal{Q} \rightarrow \Lambda^{n-k}(C^\infty(M)) \bar{\otimes} \mathcal{Q}.$$

We derive a characterization for  $(* \otimes \text{id}) : \Lambda^k(C^\infty(M)) \bar{\otimes} \mathcal{Q} \rightarrow \Lambda^{n-k}(C^\infty(M)) \bar{\otimes} \mathcal{Q}$  for all  $k = 1, \dots, n$ .

**Lemma 5.2** *Let  $\xi \in \Lambda^{n-k}(C^\infty(M)) \bar{\otimes} \mathcal{Q}$  and  $X \in \Lambda^k(C^\infty(M)) \bar{\otimes} \mathcal{Q}$ . Then the following are equivalent:*

(i) *For all  $Y \in \Lambda^k(C^\infty(M)) \bar{\otimes} \mathcal{Q}$ ,*

$$\xi \wedge Y = \langle\langle \bar{X}, Y \rangle\rangle (\text{dvol} \otimes 1_{\mathcal{Q}}) \quad (1)$$

(ii)  $\xi = (* \otimes \text{id})X$ .

*Proof:*

(i)  $\Rightarrow$  (ii):

Let  $m \in M$ . Choose a coordinate neighborhood  $(U, x_1, x_2, \dots, x_n)$  around  $x$  in  $M$  such that  $\{dx_1(m), \dots, dx_n(m)\}$  is an orthonormal basis for  $T_m^*(M)$  for all  $m \in U$ . Now for any  $l \in \{1, \dots, n\}$ , let  $\Sigma_l$  be the set consisting of  $l$  tuples  $(i_1, \dots, i_l)$  such that  $i_1 < i_2 < \dots < i_l$  and  $i_j \in \{1, \dots, n\}$  for  $j = 1, \dots, l$ . For  $I = (i_1, \dots, i_l) \in \Sigma_l$ , we write  $dx_I(m)$  for  $dx_{i_1} \wedge \dots \wedge dx_{i_l}(m)$ . Also for  $I = (i_1, \dots, i_p) \in \Sigma_p$ ,  $J = (j_1, \dots, j_q) \in \Sigma_q$ , we write  $(I, J)$  for  $(i_1, \dots, i_p, j_1, \dots, j_q)$ .

Now fix  $I \in \Sigma_k$ . Then we have a unique  $I' \in \Sigma_{n-k}$  such that

$$(* (dx_I))(m) = \epsilon(I) dx_{I'}(m),$$

where  $\epsilon(I)$  is the sign of the permutation  $(I, I')$ . Given  $X \in \Lambda^k(C^\infty(M)) \bar{\otimes} \mathcal{Q}$ , for  $m \in M$ , we have  $q_I(m) \in \mathcal{Q}$  such that

$$X(m) = \sum_{I \in \Sigma_k} dx_I(m) q_I(m).$$

Also for  $\xi \in \Lambda^{n-k}(C^\infty(M)) \bar{\otimes} \mathcal{Q}$ , we have  $w_J(m) \in \mathcal{Q}$  such that

$$\xi(m) = \sum_{J \in \Sigma_{n-k}} dx_J(m) w_J(m).$$

Hence

$$((* \otimes \text{id})X)(m) = \sum_{I \in \Sigma_k} \epsilon(I) dx_{I'}(m) q_I(m),$$

where  $I' \in \Sigma_{n-k}$  is as mentioned before.

Now we fix some  $L \in \Sigma_k$  and choose  $Y \in \Lambda^k(C^\infty(M)) \bar{\otimes} \mathcal{Q}$  such that  $Y(m) = dx_L(m) 1_{\mathcal{Q}}$ . Hence

$$(\xi \wedge Y)(m) = \sum_{J \in \Sigma_{n-k}} dx_J \wedge dx_L w_J(m).$$



But for a fixed  $L \in \Sigma_k$ , there is a unique  $J' \in \Sigma_{n-k}$  such that

$$dx_{J'}(m) \wedge dx_L(m) = \epsilon(L) d\text{vol}(m).$$

Hence

$$(\xi \wedge Y)(m) = \epsilon(L) w_{J'}(m) d\text{vol}(m).$$

On the other hand

$$\begin{aligned} & \langle \langle \bar{X}, Y \rangle \rangle (m) d\text{vol}(m) \\ &= \sum_{I \in \Sigma_k} \langle dx_I(m) q_I(m)^*, dx_L(m) 1_{\mathcal{Q}} \rangle d\text{vol}(m) \\ &= q_L(m) d\text{vol}(m). \end{aligned}$$

Hence by (3), we have  $q_L(m) = \epsilon(L) w_{J'}$ . So varying  $Y$ , we have

$$\xi(m) = \sum_{L \in \Sigma_k} \epsilon(L) dx_J(m) q_L(m),$$

i.e.  $(* \otimes \text{id})X = \xi$ . The other direction of the proof is trivial.  
□.

## 6 Smooth and inner-product preserving action

**Definition 6.1** *A topological action of  $\mathcal{Q}$  on the Fréchet algebra  $C^\infty(M)$  is called the smooth action of  $\mathcal{Q}$  on the manifold  $M$ .*

It has been shown in [12] that a smooth action  $\alpha$  of  $\mathcal{Q}$  on  $M$  extends to a  $C^*$  action on  $C(M)$  which is denoted by  $\alpha$  again.

Moreover, set  $d\alpha(df) := (d \otimes \text{id})\alpha(f)$  for all  $f \in C^\infty(M)$ . The following is proved in [12]:

**Theorem 6.2** *(i)  $d\alpha$  extends to a well defined continuous map from  $\Omega^1(C^\infty(M))$  to  $\Omega^1(C^\infty(M)) \bar{\otimes} \mathcal{Q}$  satisfying  $d\alpha(df) = (d \otimes \text{id})\alpha(f)$ . (ii) For every  $x \in M$ , the unital  $*$ -algebra  $\mathcal{Q}_x$  generated by  $(\nu \otimes \text{id})\alpha(f)(x)$ ,  $\alpha(g)(x)$  with  $f, g \in C^\infty(M)$  and all smooth vector fields  $\nu$  on  $M$ , is commutative.*

**Definition 6.3** *We call a smooth action  $\alpha$  on a Riemannian manifold  $M$  to be inner-product preserving if*

$$\langle \langle (d \otimes \text{id})\alpha(f), (d \otimes \text{id})\alpha(g) \rangle \rangle = \alpha(\langle \langle df, dg \rangle \rangle) \quad (2)$$

for all  $f, g \in C^\infty(M)$ .

In [12], it is proved that an inner product preserving action induces a canonical unitary equivariant representation on each of the bimodules of forms  $\Lambda^k(C^\infty(M))$ , to be denoted by  $d\alpha_{(k)}$ , say. Moreover, using this, smooth actions on the total

spaces of certain bundles  $E_\epsilon^k$  have been constructed in Section 8 of [12]. Let  $T_\epsilon^*(M)$  be the total space of the cotangent bundle of a compact Riemannian manifold  $M$  consisting of cotangent vectors of length less than or equal to  $\epsilon$  for some positive epsilon. The arguments of Section 8 of [12] go through verbatim to give a smooth action say  $\eta$  on  $C^\infty(T_\epsilon^*(M))$ .

$T_\epsilon^*(M)$  is a compact  $2n$  dimensional manifold. Note that  $\pi^{-1}(U) \cong U \times K$ , where  $K$  is an  $n$ -dimensional closed ball of radius  $\epsilon$ . Moreover  $T_\epsilon^*(M)$  is orientable with the following natural orientation. At the point  $(m, \omega) \in \pi^{-1}(U)$  and any choice  $\omega_1, \dots, \omega_n$  as before,  $\text{dvol}(m, \omega) \in \Lambda^{2n}(C^\infty(T_\epsilon^*(M)))$  is given by  $(\omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_n \wedge dt_1 \wedge \dots \wedge dt_n)(m, \omega)$ . It can be seen to be independent of choice of  $\omega_1, \dots, \omega_n$  and also it is non zero everywhere. Henceforth, we shall consider  $T_\epsilon^*(M)$  oriented with the globally defined non vanishing dvol as the choice of orientation.

**Lemma 6.4** *The lifted action  $\eta$  is also orientation preserving in the sense  $d\eta_{(2n)}(\text{dvol}) = \text{dvol} \otimes 1_{\mathcal{Q}}$ .*

*Proof:*

For  $m \in M$ , choose a trivializing neighborhood around  $m$  and one forms  $\omega_1, \dots, \omega_n$  such that  $\{\omega_1(x), \dots, \omega_n(x)\}$  forms an orthonormal basis for  $T_x^*(M)$  for all  $x \in U$ . Then there are  $\mathcal{Q}$ -valued functions  $f_{ij}$ 's for  $1 \leq i, j \leq n$  such that  $f_{ij}(m) \in \mathcal{Q}_m$  for all  $m \in M$  and  $d\alpha(\omega_i)(m) = \sum_j f_{ij}(m)\omega_j(m)$ . Choose and fix some smooth non-negative function  $\chi$  supported in  $U$ . By the commutativity of  $\mathcal{Q}_m$ , we get

$$\begin{aligned} & d\alpha_{(n)}(\omega_1 \wedge \dots \wedge \omega_n)(m) \\ &= \wedge_{j=1}^n \left( \sum_j f_{ij}\omega_j \right)(m) \\ &= \sum_{\sigma \in S_n} (\text{sgn } \sigma) f_{1\sigma(1)}(m) f_{2\sigma(2)}(m) \dots f_{n\sigma(n)}(m) (\omega_1 \wedge \dots \wedge \omega_n)(m) \\ &= \Delta(m)(\omega_1 \wedge \dots \wedge \omega_n)(m). \end{aligned}$$

where  $\Delta(m) = \det((f_{ij}(m)))$ . Also, we have

$$\begin{aligned} & d\eta_n(dt_1 \wedge \dots \wedge dt_n)(m, \omega) \\ &= \wedge \left( \sum_j f_{ij}(m) dt_j(m, \omega) \right) \\ &= \Delta(m)(dt_1 \wedge \dots \wedge dt_n)(m, \omega) \end{aligned}$$

Hence  $d\eta_{(2n)}(\omega_1 \wedge \dots \wedge \omega_n \wedge dt_1 \wedge \dots \wedge dt_n)(m, \omega) = \Delta(m)^2(\omega_1 \wedge \dots \wedge \omega_n \wedge dt_1 \wedge \dots \wedge dt_n)(m, \omega)$ .

Now note that

$$\begin{aligned} & \alpha(\chi)(m)^2 << d\alpha(\omega_i), d\alpha(\omega_j) >> (m) \\ &= \delta_{ij} \alpha(\chi)^2(m) \end{aligned}$$

as  $w_i$ 's are orthonormal on the support of  $\chi$ . Moreover, each  $f_{ij}(m)$  is self adjoint. Choosing any  $*$ -character  $\gamma$  on the commutative  $C^*$  algebra  $\mathcal{Q}_m$ , we see that either  $\gamma(\alpha(\chi)(m)) = 0$  or  $((\gamma(f_{ij}(m))))$  is in  $O_n(\mathbb{R})$  and its determinant  $\gamma(\Delta(m))$  is 1 or  $-1$ . Thus  $\alpha(\chi)\Delta^2 = \alpha(\chi)$ , which implies

$$d\eta_{(2n)}(\chi \text{dvol}) = \alpha(\chi)(\text{dvol} \otimes 1),$$

and hence by a partition of unity argument we complete the proof that  $\eta$  is orientation-preserving.  $\square$

## 7 Action commuting with the Laplacian , i.e. isometric

### 7.1 Isometric actions

Recall the definition of  $QISO^\mathcal{L}$  for a spectral triple satisfying certain regularity conditions from [12]. In particular, all classical spectral triples, i.e. those coming from the Dirac operator on the spinor bundle of a compact Riemannian spin manifold, do satisfy such conditions and hence  $QISO^\mathcal{L}$  is defined for them. In fact it easily follows from [13] that one can go beyond spin manifolds and define (and prove existence of) such a quantum isometry group for any compact Riemannian manifold  $M$  (without boundary) as the universal object in the category of CQG  $\mathcal{Q}$  with a faithful action  $\alpha$  on  $C(M)$  such that  $(\text{id} \otimes \phi)\alpha(C^\infty(M)) \subset C^\infty(M)$  for all state  $\phi$  and commutes with the Hodge Laplacian (to be called the  $L_2$  Laplacian)  $\mathcal{L}_2 = -d^*d$  restricted to  $L^2(M, \text{dvol})$ . We shall denote the universal object in this category by  $QISO^\mathcal{L}(M)$  in this paper. It is proved in (Theorem 3.8 of [8]) that  $QISO^\mathcal{L}(M) \cong QISO_I^+(d + d^*)$  where now  $d$  is viewed as a map on the Hilbert space of forms of all orders, i.e. the  $L^2$  closure of  $\oplus_{k=0}^{\dim} M \Lambda^k(M)$ .

Furthermore it follows from the Sobolev theorem that  $(\text{id} \otimes \phi)\alpha(C^\infty(M)) \subset C^\infty(M)$  for all state  $\phi$ . We have the following (see [12] for a proof):

**Theorem 7.1**  *$QISO^\mathcal{L}$  (and hence any subobject in the category  $\mathcal{Q}^\mathcal{L}$ ) has a smooth action on  $C^\infty(M)$ .*

Let us denote by  $\mathcal{L}$  the restriction of  $\mathcal{L}_2$  to  $C^\infty(M)$ , viewed as a Fréchet continuous operator (to be called the ‘geometric Laplacian’). When  $M$  is oriented we can also write it as  $-(\ast d)^2$ , where  $\ast$  is the Hodge  $\ast$  operator as discussed in subsection 5.2. As  $C^\infty(M)$  is a core for  $\mathcal{L}_2$ , it is clear that a CQG action  $\alpha : C(M) \rightarrow C(M) \hat{\otimes} \mathcal{Q}$  is isometric (i.e.  $(\mathcal{A}, \alpha)$  is an object in  $\mathcal{Q}^\mathcal{L}$ ) if and only if  $\alpha$  is smooth and commutes with  $\mathcal{L}$  in the sense that  $\alpha \circ \mathcal{L} = (\mathcal{L} \otimes 1)\alpha$ .

For the purpose of this paper, we need to extend the above formulation of quantum isometry group to manifolds with boundary. Choosing the Dirichlet boundary condition, we take  $d$  to be the closure of the unbounded operator with domain  $\mathcal{C} = \{f \in C^\infty(M) : f|_{\partial M=0}\}$ .

**Definition 7.2** For a compact manifold with boundary we call a smooth action  $\alpha : C(M) \rightarrow C(M) \hat{\otimes} \mathcal{Q}$  to be isometric if it maps  $\mathcal{C}$  into  $\mathcal{C} \hat{\otimes} \mathcal{Q}$  and commutes with  $\mathcal{L}_2$  on  $C^\infty(M)$ .

**Remark 7.3** For a manifold with boundary, commutation with the geometric Laplacian  $\mathcal{L}$  may not be sufficient to imply that  $\alpha$  is isometric. We also require the condition that  $\alpha(\mathcal{C}) \subset \mathcal{C} \hat{\otimes} \mathcal{Q}$ . We can prove the existence of  $QISO^\mathcal{L}$  as well as the smoothness of the action of  $QISO^\mathcal{L}$  as in [13]. It is a consequence of the fact that the Dirichlet Laplacian has discrete spectrum with finite dimensional eigen spaces and the estimate  $\|e_j(f)\|_\infty \leq C\lambda_j^{\frac{n-1}{2}} \|f\|_2$  of the eigen vectors of the Laplacian (see page 9 of [25]).

**Lemma 7.4** If  $\alpha$  commutes with the geometric Laplacian  $\mathcal{L}$  on  $\mathcal{A}$ , then  $\alpha$  is inner product preserving.

*Proof:*

$$\begin{aligned} & \langle\langle (d \otimes \text{id})\alpha(f), (d \otimes \text{id})\alpha(g) \rangle\rangle \\ &= \langle\langle df_{(0)}, dg_{(0)} \rangle\rangle \otimes f_{(1)}^* g_{(1)} \\ &= [\mathcal{L}(\overline{f_{(0)}}g_{(0)}) - \mathcal{L}(\overline{f_{(0)}})g_{(0)} - \overline{f_{(0)}}\mathcal{L}(g_{(0)})] \otimes f_{(1)}^* g_{(1)} \end{aligned}$$

On the other hand

$$\begin{aligned} & \alpha(\langle\langle df, dg \rangle\rangle) \\ &= \alpha[\mathcal{L}(\overline{f}g) - \mathcal{L}(\overline{f})g - \overline{f}\mathcal{L}(g)] \\ &= [\mathcal{L}(\overline{f_{(0)}}g_{(0)}) - \mathcal{L}(\overline{f_{(0)}})g_{(0)} - \overline{f_{(0)}}\mathcal{L}(g_{(0)})] \otimes f_{(1)}^* g_{(1)} \text{ (since } \alpha \text{ commutes with } \mathcal{L}) \end{aligned}$$

□

## 7.2 Geometric characterization of orientation-preserving isometric action

Our aim of this subsection is to prove a partial converse to the fact that an isometric action is inner product preserving. More precisely, we shall prove the following

**Lemma 7.5** Let  $N$  be an  $m$ -dimensional compact, oriented, Riemannian manifold (possibly with boundary) with  $\text{dvol} \in \Lambda^m(C^\infty(N))$  a globally defined nonzero form. Moreover let  $\eta$  be a smooth inner product preserving action on  $N$  such that  $d\eta_{(m)}(\text{dvol}) = \text{dvol} \otimes 1$ . Then  $\eta$  commutes with the geometric Laplacian.

*Proof:*

First we note that as  $\eta$  is an inner product preserving smooth action, by the results of [12] (Corollary 7.12) it lifts to an  $\alpha$ -equivariant unitary representations  $d\eta_{(k)} : \Lambda^k(C^\infty(N)) \rightarrow \Lambda^k(C^\infty(N)) \bar{\otimes} \mathcal{Q}$  for all  $k = 1, \dots, m$ . Note that without loss of generality we can replace  $\text{dvol}$  by  $\frac{\text{dvol}}{\langle\langle \text{dvol}, \text{dvol} \rangle\rangle^{\frac{1}{2}}}$  and assume that  $\langle\langle \text{dvol}, \text{dvol} \rangle\rangle = 1$ , since if  $d\eta_{(m)}$  preseves  $\text{dvol}$ , it also preserves the normalized  $\text{dvol}$ . First we claim that

$$\forall k = 1, \dots, m, d\eta_{(m-k)}(*\omega) \wedge \beta = \langle\langle \overline{d\eta_{(k)}(\omega)}, \beta \rangle\rangle (\text{dvol} \otimes 1_{\mathcal{Q}}) \quad (3)$$

$\forall \omega \in \Lambda^k(C^\infty(N)), \forall \beta \in \Lambda^k(C^\infty(N)) \bar{\otimes} \mathcal{Q}$ .

For that let  $\beta = d\eta_{(k)}(\omega')(1 \otimes q')$ . Then

$$\begin{aligned} & d\eta_{(m-k)}(*\omega) \wedge \beta \\ &= d\eta_{(m-k)}(*\omega) \wedge d\eta_{(k)}(\omega')(1 \otimes q') \\ &= d\eta_{(m)}(*(\omega) \wedge \omega')(1 \otimes q') \\ &= d\eta_{(m)}(\langle\langle \bar{\omega}, \omega' \rangle\rangle \text{dvol})(1 \otimes q') \\ &= \eta(\langle\langle \bar{\omega}, \omega' \rangle\rangle) (\text{dvol} \otimes q') \end{aligned}$$

On the other hand from unitarity of  $d\eta_{(k)}$ ,

$$\begin{aligned} & \langle\langle \overline{d\eta_{(k)}(\omega)}, d\eta_{(k)}(\omega')(1 \otimes q') \rangle\rangle \\ &= \eta(\langle\langle \bar{\omega}, \omega' \rangle\rangle) (1 \otimes q'). \end{aligned}$$

So by replacing  $\beta$  by finite sums of the type  $\sum_i d\eta_{(k)}(\omega_i)(1 \otimes q_i)$ , we can show that for  $\omega \in \Lambda^k(C^\infty(N))$  and  $\beta \in \text{Sp } d\eta_{(k)} \Lambda^k(C^\infty(N))(1 \otimes \mathcal{Q})$ ,

$$d\eta_{(m-k)}(*\omega) \wedge \beta = \langle\langle d\eta_{(k)}(\bar{\omega}), \beta \rangle\rangle (\text{dvol} \otimes 1_{\mathcal{Q}}).$$

Now, since  $\text{Sp } d\eta_{(k)}(\Lambda^k(C^\infty(N)))(1 \otimes \mathcal{Q})$  is dense in  $\Lambda^k(C^\infty(N)) \bar{\otimes} \mathcal{Q}$ , we get a sequence  $\beta_n$  belonging to  $\text{Sp } d\eta_{(k)}(\Lambda^k(C^\infty(N)))(1 \otimes \mathcal{Q})$  such that  $\beta_n \rightarrow \beta$  in the Hilbert module  $\Lambda^k(C^\infty(N)) \bar{\otimes} \mathcal{Q}$ .

But we have

$$d\eta_{(m-k)}(*\omega) \wedge \beta_n = \langle\langle d\eta_{(k)}(\bar{\omega}), \beta_n \rangle\rangle (\text{dvol} \otimes 1_{\mathcal{Q}}).$$

Hence the claim follows from the continuity of  $\langle\langle, \rangle\rangle$  and  $\wedge$  in the Hilbert module  $\Lambda^k(C^\infty(N)) \bar{\otimes} \mathcal{Q}$ .

Combining Lemma 5.2 and (3) we immediately conclude the following:

$$d\eta_{(m-k)}(*\omega) = (* \otimes \text{id})d\eta_{(k)}(\omega) \text{ for } k \geq 0. \quad (4)$$

Now we can prove that  $\eta$  commutes with the geometric Laplacian of  $N$ . For  $\phi \in C^\infty(N)$ ,

$$\begin{aligned}
& \eta(*d*d\phi) \\
&= (*\otimes \text{id})d\eta_{(m)}(d*d\phi) \text{ (by equation (4) with } k=m\text{)} \\
&= (*d\otimes \text{id})d\eta_{(m-1)}(*d\phi) \\
&= (*d\otimes \text{id})(*\otimes \text{id})d\eta(d\phi) \text{ (again by equation (4))} \\
&= (*d\otimes \text{id})(*d\otimes \text{id})\eta(\phi) \\
&= ((*d)^2\otimes \text{id})\eta(\phi).
\end{aligned}$$

□

## 8 Application: Non existence of genuine CQG action

### 8.1 The stably parallelizable case

We now introduce the notion of stably parallelizable manifolds.

**Definition 8.1** *A manifold  $M$  is said to be stably parallelizable if its tangent bundle is stably trivial.*

We recall the following from [21]:

**Proposition 8.2** *A manifold  $M$  is stably parallelizable if and only if it has trivial normal bundle when embedded in a Euclidean space of dimension higher than twice the dimension of  $M$ .*

**Proof:** see discussion following the Theorem (7.2) of [14].

□

We note that parallelizable manifolds (i.e. which has trivial tangent bundles) are in particular stably parallelizable. Moreover, given any compact Riemannian manifold  $M$ , its orthonormal frame bundle  $O_M$  is parallelizable. Also given any stably parallelizable manifold  $M$ , the total space of its cotangent bundle is again stably parallelizable.

Recall the manifold  $T_\epsilon^*(M)$  for a smooth compact manifold  $M$ . If  $M$  is isometrically embedded in some  $\mathbb{R}^N$ , then we consider the set  $V = \{(u, v) : u \in \mathbb{R}^N, v \in \mathbb{R}^N \text{ such that } \|v\| \leq \epsilon\}$ , then we define  $\Phi : T_\epsilon^*(M) \rightarrow V$  by  $\Phi(m, v) = (\phi(m), d\phi(v))$  where  $\phi$  is the isometric embedding of  $M$ . Then it is easy to see that  $\Phi(T_\epsilon^*(M))$  is a submanifold of  $V$  and in fact it is a neat submanifold of  $V$ . So by Theorem 6.3 (page 114) of [18], we have

**Lemma 8.3**  *$T_\epsilon^*(M)$  has a tubular neighborhood for some  $\delta > 0$  in  $V$ . It is denoted by  $\mathcal{N}_\delta(T_\epsilon^*(M))$ .*

As  $T_\epsilon^*(M)$  has a trivial normal bundle in  $V$ , the tubular neighborhood is actually diffeomorphic to  $T_\epsilon^*(M) \times B_\delta^{2(N-n)}$ , where  $n$  is the dimension of the manifold. We denote the global coordinates for  $B_\delta^{2(N-n)}$  by  $u_1, \dots, u_{2(N-n)}$ . Let us recall from [12] the lift of a smooth action on a compact, stably parallelizable, Riemannian manifold to its tubular neighborhood. We denote the lift of the action  $\eta$  on the manifold  $T_\epsilon^*(M)$  to  $\mathcal{N}_\delta(T_\epsilon^*(M))$  by  $\eta'$ . Also we take the canonical volume form of  $\mathcal{N}_\delta(T_\epsilon^*(M))$  to be  $\text{dvol} \otimes du_1 \wedge \dots \wedge du_{2(N-n)}$ , where  $\text{dvol}$  is the volume form of  $T_\epsilon^*(M)$ . Then we have

**Proposition 8.4** (i)  $\eta'$  is inner product preserving.  
(ii)  $\eta'$  is orientation preserving.

*Proof:*

The first statement was proved in [12] (Lemma 9.3). For the second statement it is enough to observe that  $\eta'(u_i) = u_i$  for all  $i = 1, \dots, 2(N-n)$  and for the functions of the form  $f \circ \pi$ , where  $\pi$  is the projection of the normal bundle,  $\eta'(f \circ \pi)(y) = \eta(f)(\pi(y))$  for all  $y \in \mathcal{N}_\delta(T_\epsilon^*(M))$ , which follows from the definition of the extension  $\eta'$ . and the fact that  $\eta$  is orientation preserving.  $\square$

Let  $\{y_i : i = 1, \dots, N\}$  be the standard coordinates for  $\mathbb{R}^N$ . We will also use the same notation for the restrictions of  $y_i$ 's if no confusion arises.

**Definition 8.5** A twice continuously differentiable, complex-valued function  $\Psi$  defined on a non empty, open set  $\Omega \subset \mathbb{R}^N$  is said to be harmonic on  $\Omega$  if

$$\mathcal{L}_{\mathbb{R}^N} \Psi \equiv 0,$$

where  $\mathcal{L}_{\mathbb{R}^N} \equiv \sum_{i=1}^N \frac{\partial^2}{\partial y_i^2}$ .

**Lemma 8.6** Let  $W$  be a manifold (possibly with boundary) embedded in some  $\mathbb{R}^N$  and  $\{y_i\}$ 's for  $i = 1, \dots, N$ , be the coordinate functions for  $\mathbb{R}^N$  restricted to  $W$ . If  $W$  has non empty interior in  $\mathbb{R}^N$ , then  $\{1, y_i y_j, y_i : 1 \leq i, j \leq N\}$  are linearly independent, i.e.  $\{1, y_1, \dots, y_N\}$  are quadratically independent.

We call any action which preserves  $V = \{1, y_1, \dots, y_N\}$  affine.

**Lemma 8.7** Let  $\Phi$  be a smooth action of a CQG on a compact subset of  $\mathbb{R}^N$  which commutes with  $\mathcal{L}_{\mathbb{R}^N}$ , Then  $\Phi$  is affine i.e.

$$\Phi(y_i) = 1 \otimes q_i + \sum_{j=1}^N y_j \otimes q_{ij}, \text{ for some } q_{ij}, q_i \in \mathcal{Q},$$

for all  $i = 1, \dots, N$ , where  $y_i$ 's are coordinates of  $\mathbb{R}^N$ .

*Proof:*

As  $\Phi$  commutes with the geometric Laplacian and  $\mathcal{L}_{\mathbb{R}^N} \frac{\partial}{\partial y_j} = \frac{\partial}{\partial y_j} \mathcal{L}_{\mathbb{R}^N}$ ,  $\mathcal{L}_{\mathbb{R}^N} y_j =$

0 for all  $j$ , we get

$$\begin{aligned}
& (\mathcal{L}_{\mathbb{R}^N} \otimes \text{id}) \left( \frac{\partial}{\partial y_j} \otimes \text{id} \right) \Phi(y_i) \\
&= \left( \frac{\partial}{\partial y_j} \otimes \text{id} \right) \Phi(\mathcal{L}_{\mathbb{R}^N} y_i) \\
&= 0.
\end{aligned}$$

Let  $D_{ij}(y) = ((\frac{\partial}{\partial y_i} \otimes \text{id})\Phi(y_j))(y)$ . Note that as  $d\Phi$  is an  $\Phi$ -equivariant unitary representation, by Lemma 3.8  $((D_{ij}(y)))_{i,j=1,\dots,N}$  is unitary for all  $y \in W$ . Pick  $y_0$  in the interior of  $W$  (which is non empty). Then the new  $\mathcal{Q}$  valued matrix  $((G_{ij}(y))) = ((D_{ij}(y))((D_{ij}(y_0)))^{-1})$  is unitary (since  $D_{ij}(y)$  is so).  $G_{ij}(y)$  is unitary for all  $y \Rightarrow |\psi(G_{ij}(y))| \leq 1$  And  $|\psi(G_{ii}(y_0))| = 1$ .  $\psi(G_{ii}(y))$  is a harmonic function on an open *connected* set  $\text{Int}(W)$  which attains its supremum at an interior point. Hence by corollary 1.9 of [1] we conclude that  $\psi(G_{ii}(y)) = \psi(G_{ii}(y_0)) = 1$ .  $((G_{ij}(y)))$  being unitary for all  $y$ , we get  $G_{ij} = \delta_{ij} \cdot 1_{\mathcal{Q}}$ . Then  $((D_{ij}(y))((D_{ij}(y_0)))^{-1} = 1_{M_N(\mathcal{Q})}$ , i.e.  $((D_{ij}(y))) = ((D_{ij}(y_0)))$  for all  $y \in W$ . Hence  $\Phi$  is affine with  $q_{ij} = D_{ij}(y_0)$   $\square$

We also state the following Lemma without proof. For the proof reader might see [12].

**Lemma 8.8** *Let  $\mathcal{C}$  be a unital commutative  $C^*$  algebra and  $x_1, x_2, \dots, x_N$  be self adjoint elements of  $\mathcal{C}$  such that  $\{x_i x_j : 1 \leq i \leq j \leq N\}$  are linearly independent and  $\mathcal{C}$  be a unital  $C^*$  algebra generated by  $\{x_1, x_2, \dots, x_N\}$ . Let  $\mathcal{Q}$  be a compact quantum group acting faithfully on  $\mathcal{C}$  such that the action leaves the span of  $\{x_1, x_2, \dots, x_N\}$  invariant. Then  $\mathcal{Q}$  must be commutative as a  $C^*$  algebra, i.e.  $\mathcal{Q} \cong C(G)$  for some compact group  $G$ .*

**Remark 8.9** *This is the only place where we need the manifold to be connected.*

$\square$

**Corollary 8.10** *Let  $M$  be a smooth, compact, orientable, connected, stably parallelizable manifold. Then if  $\alpha$  is a faithful smooth action of a CQG  $\mathcal{Q}$ . Then  $\mathcal{Q}$  must be commutative as a  $C^*$  algebra i.e.  $\mathcal{Q} \cong C(G)$  for some compact group  $G$ .*

*Proof:*

First recall from [12] (Theorem 7.13) that given a smooth action  $\alpha$  of a CQG  $\mathcal{Q}$  on a compact Riemannian manifold  $M$ , we can equip the manifold with a Riemannian structure such that the action becomes inner product preserving. So, by applying the averaging trick we reduce the action to an inner product preserving action first. Then we lift the action to the total space of the cotangent bundle. By Lemma 6.4, the action is also orientation preserving. Now again using the averaging trick we equip the total space of the cotangent bundle with a new Riemannian metric such that the action is inner product preserving. Then



we lift this orientation preserving and inner product preserving action to the tubular neighborhood of the total space of the cotangent bundle (which exists by Lemma 8.3) and by Proposition 8.4, we see that it is still orientation and inner product preserving. So by lemma 7.5, it commutes with the geometric Laplacian of the tubular neighborhood, which is an open subset of  $\mathbb{R}^N$  for some  $N$ . Now by applying Lemma 8.7, Lemma 8.6 and Lemma 8.8, we complete the proof.  $\square$

Using the above result and using the isometric lift of an action on a manifold to the total space of its orthonormal frame bundle (which is parallelizable) we get the main result of [12] which states that

**Theorem 8.11** *Let  $\alpha$  be a smooth, faithful action of a CQG  $\mathcal{Q}$  on a compact, connected smooth manifold  $M$ . Then  $\mathcal{Q}$  must be commutative as a  $C^*$  algebra i.e.  $\mathcal{Q} \cong C(G)$  for some compact group  $G$ .*

**Corollary 8.12** *The quantum isometry group of a compact, connected, Riemannian manifold coincides with the classical isometry group of the manifold.*

*Proof:*

Follows from the fact (Theorem 7.1) that an isometric action of a compact quantum group is smooth.

$\square$

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